Quantum Entanglement: An Exploration of a Weird Phenomenon[1]

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Abstract: In this paper, quantum entanglement, which is a quantum mechanical phenomenon where particles can have influence on others regardless of the distance between them, is deeply explored. A historical context is first presented to trace the different events that led to quantum mechanics as a theory; its mathematical formalism is then presented to help understanding the theoretical aspect of quantum entanglement. Later, the validity of quantum entanglement is explored where arguments from Einstein, Bell, and Neumann are presented and where experiments from Clauser, Holt, Shimony, and Horne; and Aspect, Grangier, and Roger are presented.

[1] I would like to thank Professor Carlos Ramos for willing to work with me on this paper, Robert Slattery for proofreading it, and the reviewers for their useful suggestions.

"God does not play dice," confidently affirmed Albert Einstein in his fight against quantum mechanics. Indeed, Einstein did not appreciate the stochastic nature of quantum mechanics that represented a threat to determinism, which had been the ultimate criterion for science. However, over the years, quantum mechanics has shown that it is the best candidate to describe the subatomic world even though its phenomena contradict humans' most intuitive understanding of their physical world. One of those phenomena is quantum entanglement, where two particles, after some interaction, have some influence on each other regardless of the distance between them, distance that can theoretically be light-years between them. In spite of its unorthodoxical nature, quantum entanglement is well explained by physics and is shown to be consistent with experiments. One can then understand why this spectacular phenomenon is worth exploring; to do so, a historical background of quantum mechanics will be presented to better put the subject matter into context followed by the mathematical formalism of quantum mechanics, the theoretical explanation of entanglement, and some experiments that confirm the phenomenon.

Historical Background of Quantum Mechanics

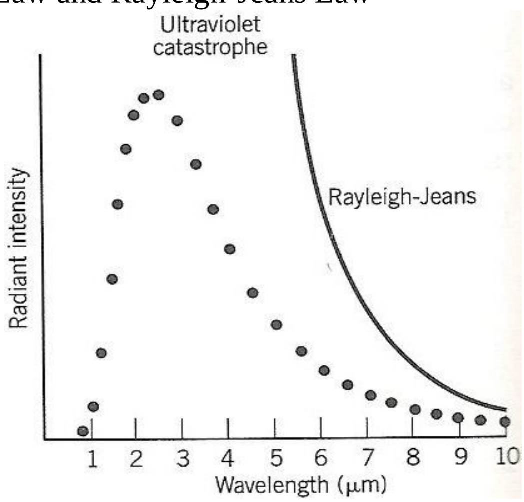
The Blackbody Radiation Problem

The birth of quantum mechanics is considered to take place when Max Planck (1858-1947) gave his original explanation about the black body radiation problem, which could not be explained by classical physics. A black body can be considered as an ideal object that can emit all the energy that it absorbs. The explanation provided by the classical mechanics about how objects emit energy was not in accord with what really happened; it stipulated that a hot object would radiate within the entire wave spectrum and that the intensity of the radiation would be infinitely bigger (Aczel, 2001, p. 34). Indeed, the Rayleigh-Jeans law, which is valid only for long wavelengths, states that

$$R_V = \frac{2v^2 kT}{c^2},$$

where ν is frequency, k is the Boltzmann constant, T is temperature, and c is the speed of light; one can easily see that the intensity of the radiation or radiance will go to infinity when the frequency goes to infinity[2] (Weisstein, 2007).

Figure 1.
Difference between Planck Law and Rayleigh-Jeans Law



Note. This graph is taken from *Modern Physics* by K. Kenneth (1983), p.64.

Because of the incapacity of the theory to give a satisfactory explanation when wavelengths are very short, a new theory needs to be taken into account. Despite its counter-intuitiveness, Planck's law happens to give a precise explanation of what really happens: Instead of considering energy as continuous, Max Planck hypothesized that energy is emitted in packets or quanta and that the energy of a quantum is given by

$$E = hv,$$

where h is the Planck's constant ($h = 6.6262 \times 10^{-34}\text{joules} - \text{seconds}$) (Aczel, 2001, p. 35). From his hypothesis, he arrived at his famous law[3], which gives the spectrum of radiant intensity with respect to wavelength:

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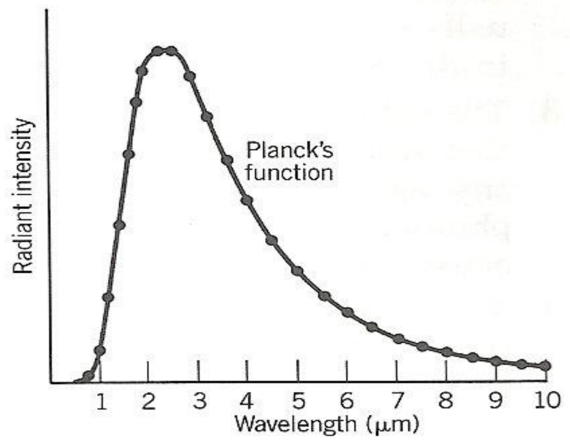
[2] This is also known as the "ultraviolet catastrophe."

$$R(\lambda) = \left(\frac{c}{4}\right)\left(\frac{8\pi}{\lambda^4}\right)\left[\left(\frac{hc}{\lambda}\right)\frac{1}{e^{\frac{hc}{\lambda kT}} - 1}\right],$$

where $R(\lambda)$ is the radiant intensity[4], h is the Planck constant, λ is wavelength, c is the speed of light, k is the Boltzmann's constant, and T is absolute temperature in Kelvin (Krane, 1983, p. 65).

Figure 2.
Planck's Law



Note. This graph is taken from *Modern Physics* by K. Kenneth (1983), p. 65.

One can see that the radiant intensity approaches zero when the value of the wavelength goes to infinity and that the radiant reaches a maximum when it gets smaller, that which was predicted by the Rayleigh-Jeans law.

Although Planck thought that his discovery was simply a mathematical convenience to help explain the black body radiation problem, his idea was even deeper than he imagined, for it helped to explain more physical phenomena that still could not be explained by classical physics. One of the most famous phenomena is the photoelectric effect, which happens when light of a specific frequency can eject electrons from a metallic plate (Penrose, 2005, p. 501). Based on Maxwell's electromagnetic wave theory, the energy of the electrons emitted is proportional to

[3] A more detailed elaboration on Planck's law can be found in *The Conceptual Development of Quantum Mechanics* by Max Jammer.
[4] It is a measure of the intensity of electromagnetic radiation.

the intensity of the incident light, but this does not happen in reality; although more electrons are ejected with increasing intensity of the light, the energy of each electron does not increase (Penrose, 2005, p. 502). However, the energy of each electron is proportional to the frequency of light. This led Einstein[5] to hypothesize that light is also made of particles, called photons, with an energy $E = h\nu$ for each photon; h is the Planck constant (Penrose, 2005, p. 502). In 1923, Millikan was awarded a Nobel Prize for his experiment proving the photoelectric effect (Jammer, 1966, p. 36).

This radical discovery revived the perpetual debate about the nature of light, which had started hundreds of years before among Newton, Laplace, Biot, Foucault, and Breguet (Jammer, 1966, p. 31). After the new finding about the particle nature of light, de Broglie proposed the radical hypothesis that if waves have particle properties, particles also have wave properties; this hypothesis is well known as the wave-particle duality. Indeed, de Broglie started with Einstein's famous formula for energy to arrive at his conclusion that particles can have wave properties; he so proceeds

$$E = mc^2,$$

m is for the mass of a particle and c is the speed of light.

The energy can also be written as

$$E = \sqrt{p^2c^2 + m_0^2c^4};$$

for rest mass equal to zero, we have

$$E = pc \Rightarrow p = \frac{E}{c}$$

[5] Einstein explained the photoelectric effect in his paper of 1905, "On a Heuristic Viewpoint Concerning the Production and Transformation of Light." For this contribution, he was awarded the 1921 Nobel Prize (Penrose, 2005, p. 501).

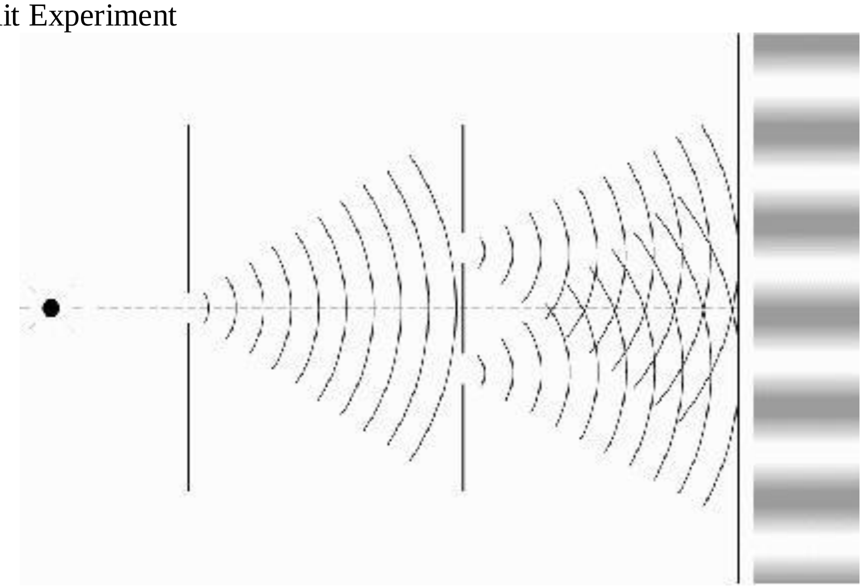
$$\Rightarrow p = \frac{h\nu}{c} = \frac{hc}{\lambda c} = \frac{h}{\lambda}$$

$$\Rightarrow \lambda = \frac{h}{p} \text{ (Nave, n. d.)} .$$

One can see that this equation links the notion of wavelength to the notion of momentum that is attributed to a particle. Besides, the reason why wavelength of big objects cannot be seen is because the value of the wavelength is very small by the value of h, the Planck constant, and the relatively big value of the momentum.

After de Broglie's hypothesis, for which he was awarded the Nobel Prize, several experiments proved its validity, and one of those experiments was the famous Two-Slit Experiment that was first performed by Thomas Young; this experiment originally showed the wave nature of light that showed an interference pattern on a wall when passing through two slits (Aczel, 2001, p. 18). Later in 1989, Tonomura and others performed the same experiment with an electron passing through the two slits, and the same interference pattern is obtained (Aczel, 2001, p. 18).

Figure 3.
The Double-Slit Experiment



Note: This graph is taken from U. Mallik's webpage
http://www.physics.uiowa.edu/~umallik/adventure/quantumwave/02kumar_yds.jpg.

Mathematical Foundation of Quantum Mechanics

Because of the systematic difference between classical physics and quantum physics, physicists found it necessary to develop a theory that does not depend on classical physics results, as it was the case to explain quantum mechanical results with classical physical laws. As a result, Heisenberg developed his then obscured matrix mechanics (Jammer, 1966, pp. 197-200). Later, Schrödinger developed wave mechanics, which was more appreciated among physicists than the matrix mechanics. However, Paul Dirac reconciled both formalisms by showing that they are equivalent in his well-known book, *The Principles of Quantum Mechanics* (Penrose, 2005, p. 538).

Before we go into details about quantum entanglement, which is our subject matter, we need to introduce the mathematical formalism of quantum mechanics, which is essential to understand the entanglement phenomenon. To do this, we will mostly adopt the Dirac and von Neumann's mathematical formulations.

In this section, we will present the five fundamental postulates of quantum mechanics along with essential definitions of key terms that will be used later to explain quantum entanglement; this section comes mainly from the lecture notes of Professor Martin Plenio (2002) of Imperial College London:

Postulate 1: *The state of a quantum system is described by a vector in a Hilbert space H.*

A quantum state is a mathematical object that describes a quantum system, which is a physical system at the microscopic level. It needs to be noted that vectors used in this sense are not the usual vectors[6] with an origin and an endpoint. These are abstract mathematical objects that are members of a complex vector space:

[6] By the way, the sets of all vectors with an origin and an endpoint also form a vector space.

Definition: *Given a quadruple (V, C, +, ·) where a V is a set of vectors, C denotes the set of complex numbers, + denotes the group operation of addition, and · denotes the multiplication of a vector with a complex number. (V, C, +, ·) is called a complex vector space iff*

(V, +) is an Abelian group, which means that

1. $\forall\, |a\rangle, |b\rangle \in V \Rightarrow |a\rangle + |b\rangle \in V.$
2. $\forall\, |a\rangle, |b\rangle, |c\rangle \in V \Rightarrow (|a\rangle + \, |b\rangle) + |c\rangle = |a\rangle + (|a\rangle + \, |c\rangle).$
3. $\exists |0\rangle \in V\ so\ that\ \forall |a\rangle \in V \Rightarrow |a\rangle + |0\rangle = |a\rangle.$
4. $\forall\, |a\rangle \in V: \exists(-|a\rangle)\ so\ that\ \ (-|a\rangle) + |a\rangle = |0\rangle.$
5. $\forall\, |a\rangle, |b\rangle \in V \Rightarrow |a\rangle + |b\rangle = |b\rangle + |a\rangle.$

The multiplication satisfies

1. $\forall \alpha \in C, \forall\, |a\rangle \in V \Rightarrow \alpha|a\rangle \in V.$
2. $\forall\, |a\rangle \in V \Rightarrow 1 \cdot |a\rangle = |a\rangle.$
3. $\forall \alpha, \beta \in C, \forall\, |a\rangle \in V \Rightarrow (\alpha \cdot \beta)\ \cdot |a\rangle = \alpha \cdot (\beta \cdot |a\rangle).$
4. $\forall \alpha, \beta \in C, \forall\, |a\rangle, |b\rangle \in V \Rightarrow \alpha \cdot (|a\rangle + |b\rangle) = \alpha \cdot |a\rangle + \alpha \cdot |b\rangle\ and$

   $(\alpha + \beta) \cdot |a\rangle = \alpha \cdot |a\rangle + \beta|a\rangle.$

Definition: *A vector space H is a Hilbert space if*

H is a unitary vector space[7]*, and*

H is complete[8]*.*

The choice for the state space to be a Hilbert space has logical reasons. According to the superposition principle[9], some states will overlap in other states, which make them non-

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[7] Vector spaces on which a scalar product is defined.

[8] A vector space V is complete if every Cauchy sequence of elements from the vector space V converges toward an element of V.

[9] This idea will be well understood when we introduce the wave interpretation of quantum states. The principle states, "the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus."

orthogonal, so the notion of to what extent they are overlapped can be described with scalar product. Also, the space has to be complete because the infinite summation of the states that describes a state has to give a physical state; in other terms, the infinite series has to converge to an element of the vector space (Plenio, 2002, p. 40).

Postulate 2: *Observable quantum mechanical quantities*[10] *are described by Hermitian operators*[11] $\hat{A}$ *on the Hilbert space H. The eigenvalues* a_i *of the Hermitian operator are the possible measurement results.*

Definition: *A linear operator* $\hat{A} : H \rightarrow H$ *associates to every vector* $|a\rangle \in H$ *a vector* $\hat{A}|a\rangle \in H$ *such that*

$$\hat{A}(\alpha|a\rangle + \beta|b\rangle) = \alpha\hat{A}|a\rangle + \beta\hat{A}|b\rangle,$$

for all $|a\rangle, |b\rangle \in H$ *and* $\alpha, \beta \in C$.

Before we give the definition of Hermitian operator, let us define adjoint operator first:

Definition: *The adjoint operator* $\hat{A}^{\dagger}$[12] *corresponding to the linear operator* $\hat{A}$ *is the operator such that for all* $|a\rangle, |b\rangle$

$$\langle b|\hat{A}^{\dagger}|a\rangle \equiv \langle a|\hat{A}^{\dagger}|b\rangle^{*}\,[13].$$

Definition: *An operator is* $\hat{A}$ *is called Hermitian or self-adjoint if it is equal to its adjoint operator.*

Hermitian operators are ideal candidates to describe observables because they have convenient properties: all eigenvalues for a Hermitian operator are real, and eigenvectors to different eigenvalues are orthogonal. The eigenvalues correspond to measurement outcomes that

[10] Observables are the physical quantities that can be measured, such as the position, momentum, and energy of a system.
[11] An operator is a mathematical entity that transforms an element or a group of elements of a set into another element or another group of elements of the same set; for example, the derivative operator transforms a function into another function.
[12] It is read "A dagger."
[13] The asterisk stands for complex conjugation.

must be real to express meaningful result of a physical system; also, different outcomes will correspond to different states of the system (Plenio, 2002, p. 47).

Now that we know that physical quantities are described by Hermitian operators, we will define the five fundamental operators used in quantum mechanics: position, momentum, time, energy, and density operators. Before we proceed, we need to highlight that we will use complex-valued functions[14] of real variables as vectors when defining those operators because it is more convenient. To show that a function is a vector in the Hilbert space is to show that it follows the requirements for a vector to be in the Hilbert space.

Position Operator

The position operator gives the position of a particle in space; usually, the operator will act on a wave-function ψ[15], which describes the state of a quantum system. Let us see how the position operator can be derived using Dirac notation: considering the ket vector $|\varphi\rangle$ that can be written in a basis $\{|e\rangle_i\}$, the i-th component of the ket vector can be written with the complex inner product[16] $\langle e_i|\varphi\rangle$. So, the ket vector can be written as $|\varphi\rangle = \langle e_1|\varphi\rangle + \cdots + \langle e_n|\varphi\rangle$. This notation can be extended to the inner product of any two vectors in the Hilbert space; for example

$$\langle\gamma|\varphi\rangle = \sum_i \langle\gamma|e_i\rangle\langle e_i|\varphi\rangle\,;$$

with integration, this inner product can be expressed in that way:

[14] If a function can be represented by an infinite series that is convergent and if it is square-integrable, that is $\int_{-\infty}^{\infty} |f(x)|^2\,dx$ exists and is finite, it can be considered as a vector in an infinite dimensional vector space.

[15] Ψ is the probability amplitude for different configurations of a system. This function is in the famous Schrödinger's equation. For general quantum systems, the form of the equation is $i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = H\Psi(\boldsymbol{r},t)$, where $i\hbar\frac{\partial}{\partial t}$ is the momentum operator and H the Hamiltonian (energy operator). For a single particle moving in a potential V, the form of the equation is $i\hbar\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = -\frac{\hbar^2}{2m}\nabla^2\Psi(\boldsymbol{r},t) + V.\Psi(\boldsymbol{r},t)$. For a derivation of this equation, one can consult *The Conceptual Development of Quantum Mechanics* by Max Jammer and *Philosophic Foundations of Quantum Mechanics* by Hans Reichenbach.

[16] This notion can be compared the inner product of a basis vector and a vector written in this basis in a real vector space.

$$\langle\gamma|\varphi\rangle = \int_{-\infty}^{\infty} dx\gamma * (x)\varphi\,(x).$$

Considering $|x\rangle$ to be a vector basis for the function space, the integral can be written with the Dirac notation:

$$\langle\gamma|\varphi\rangle = \int \langle\gamma|x\rangle\,\langle x|\varphi\rangle dx.$$

Let us now see how the position can be written with the Dirac notation. By definition, the position expectation value is given by

$$\begin{aligned}\langle\psi|\hat{x}|\psi\rangle &\equiv \int x|\psi(x)|^2\,dx \\ &= \int \langle\psi|x\rangle x\langle x|\psi\rangle\,dx \\ &= \langle\psi|\left(\int x|x\rangle\langle x|\,dx\right)|\psi\rangle^{17}.\end{aligned}$$

So,

$$\hat{x} = \int x|x\rangle\langle x|\,dx,$$

where $\int |x\rangle\langle x|\,dx$ is the identity operator.

Momentum Operator

The momentum operator gives the momentum of the state of a quantum system; by definition,

$$\langle x|\hat{p}|\psi\rangle = \int \langle x|p\rangle\,p\langle p|\psi\rangle dp$$

$$\hat{p} = \int p|p\rangle\langle p|dp;$$

[17] The reason why $\langle\psi|$ can be taken out of the integral is because it acts as a delta function, which is called an improper wave-function since delta functions are not square integrable. Let us see why $\langle\psi|$ acts as a delta function: if we define a linear functional $\langle x_o|$ by $\langle x_o|\psi\rangle \equiv \psi(x_0)$, the inner product can be defined by $(|x_0\rangle, |\psi\rangle) \equiv \langle x_o|\psi\rangle = \psi(x_0)$. Now, we need to know the relationship between the bra $\langle x_o|$ and ket $|x_0\rangle$ and what wave-function $\delta^*_{x_0}(x)$ it corresponds to; to so, we write the inner product of the two vectors with integral, which is $\int_{-\infty}^{\infty} \delta^*_{x_0}(x)\psi(x)dx =$ *x0,ψ≡xoψ=ψx0*. This is possible when *δx0∗x* acts a delta function; thus, in general, the bra vector in an inner product acts as a delta function, which can be taken out of the integral. (Plenio, 2002, p. 59).

the differential operator equivalent to this operator is $\frac{\hbar}{i}\nabla$.

Energy Operator

The energy operator, also called Hamiltonian, gives the total energy of a system. By the law of conservation of energy, the total energy of a system is the sum of its kinetic energy and its potential energy. In terms of language of differential operators, the kinetic energy operator[18] in three dimensions is $-\frac{\hbar^2}{2m}\nabla^2$ and the potential energy operator is just V(x, y, z), so the energy operator is

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z)(\text{Penrose}, 2005, \text{p}. 497).$$

Density Operator

The density operator helps to determine the expectation value[19] of an observable for an ensemble of states, which is called a mixed state. The expectation value of the observable Â is then

$$\langle\hat{A}\rangle = \sum_i p_i \, tr\{\hat{A}|\psi_i\rangle\langle\psi_i|\}$$

$$= tr\left\{\hat{A}\sum_i p_i|\psi_i\rangle\langle\psi_i|\right\}$$

$$= tr\{\hat{A}\rho\}^{20}$$

$$\Rightarrow \rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|\text{, which is the density operator.}$$

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[18] The kinetic energy operator stems from Newtonian concept of kinetic energy. In three dimensions, kinetic energy K is given by $K = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) = \frac{1}{2m}(m^2v_x^2 + m^2v_y^2 + m^2v_z^2)$. Knowing that momentum *p* = *m v*, we then have $K = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) = \frac{1}{2m}\left(\frac{\hbar^2}{i^2}\frac{\partial^2}{\partial x^2} + \frac{\hbar^2}{i^2}\frac{\partial^2}{\partial y^2} + \frac{\hbar^2}{i^2}\frac{\partial^2}{\partial z^2}\right) = -\frac{\hbar^2}{2m}\nabla^2$.

[19] In quantum mechanics, the expectation value of an operator is the average value of the outcomes in an experiment that takes place indefinitely (Phillips, 2003, p. 36).

[20] The trace of an operator Â on N dimensional Hilbert space is defined as $tr(\hat{A}) = \sum_{i=1}^{N}\langle\psi_i|\hat{A}|\psi_i\rangle$ for any orthonormal set of basis vectors $\{\psi_i\}$.

Some important properties will be given in the next theorem whose proof can be found in Plenio's Lectures:

Theorem: *Any density operator satisfies*

ρ is a Hermitian operator.

ρ is a positive semi-definite operator, that is $\forall|\psi\rangle$: $\langle\psi|\rho|\psi\rangle \geq 0$.

$$tr\{\rho\} = 1.$$

We will use the concept of density operator when we will later develop entangled states.

Postulate 3: *The state of a quantum mechanical system after the measurement of general observable $\hat{A}$ with the result being the possibly degenerate eigenvalues*[21] *a_i is given by*

$$P_i|\psi\rangle,$$

Where P_i is the projection operator on the subspace of H spanned by all the eigenvectors of $\hat{A}$ with eigenvalues a_i, i.e.

$$P_i = \sum_i |\psi_i\rangle\langle\psi_i|.$$

The notion of measurement is essential to quantum mechanics and therefore has very rigorous theory behind it. According to the first postulate, the state of the quantum system is a vector, and, according to the second postulate, observables are linear operators that are applied on the vectors and the possible measurement outcomes are given by the eigenvalues. Indeed, care needs to be taken when dealing with the eigenvalues because the situation gets complicated when there are several eigenvalues for one operator. Before we go any further, let us see some properties of eigenvalues and eigenvectors. By definition, when an operator H acts on a vector u to give a constant λ times u, λ is called an eigenvalues and u an eigenvector (Byron, Jr. & Fuller, 1992, p. 120). To determine λ and u of H, we can use the matrix equation

[21] Degenerate eigenvalues are eigenvalues that has more than one linearly independent eigenvector.

$$Hu = \lambda u \Rightarrow (H - \lambda I)u = 0,$$

where I is the identity matrix. To have a value for u, there are two possibilities:

If $(H - \lambda I)$ is nonsingular[22], u has to be zero, which does not give a useful information about u.

If $(H - \lambda I)$ is singular, its determinant is zero; we would then have this characteristic equation[23] for H:

$$\det((H - \lambda I)) = \det\begin{bmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - \lambda \end{bmatrix} = 0.$$

So, the eigenvalues of the H are the n roots of the characteristic polynomial, which are in the complex field[24] (Byron, Jr. & Fuller, 1992, p. 120). As a result, the projector operator in postulate 3 deals with the multiplicity of eigenvalues that an observable may have by helping to obtain a result that makes sense:

Definition: *An operator P is called a projector if it satisfies*

$$P = P^{\dagger}$$

$$P = P^2.$$

Lemma: The *eigenvalues of a projection operator can only have the values of 0 or 1.*

Postulate 4: *The probability of obtaining the eigenvalue a_i in a measurement of the observable $\hat{A}$ is given by*

$$p_i = \|P_i|\psi\rangle\|^2 = \langle\psi|P_i|\psi\rangle.$$

On can then see why the eigenvalue of the projector operator cannot exceed zero; otherwise, the probability would not make sense.

[22] A matrix is nonsingular iff its determinant is different from zero (Weisstein, 2009, n.p.).
[23] $\det(H - \lambda I) = 0$
[24] This is one of the reasons the quantum state space is a required to be complex.

Postulate 5: The time evolution of the quantum state of an isolated quantum mechanical system is determined by the Schrödinger equation[25]

$$i\hbar\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

where H is the Hamiltonian operator of the system.

Quantum Entanglement

Entanglement, first coined by Schrödinger as "Verschränkung" in 1935, is a phenomenon where a strong correlation exists between subsystems of a compound state, regardless of the distance between them (Horodecki, 2007, p. 3). For example, in an experiment, if two particles are entangled, the measurement on one particle simultaneously affects the other one regardless of the distance between the two particles. Let us now see the mathematical formulation of entanglement states before we explain the physical existence of them. Entangled states can be understood as states that cannot be written as product states; let us see what product states are. In the mathematical foundation of quantum mechanics above, we considered systems of one particle; we now will consider systems of two particles, which can be generalized to system of several particles. Let us consider particles A and B in the Hilbert spaces H_A and H_B spanned by the sets of basis states $\{|\varphi_i\rangle\}_{i=1,\dots,N}$ and $\{|\psi_j\rangle\}_{j=1,\dots,N}$ respectively; the state $\Psi_{AB}$of the composite system formed by the two particles is given by the tensor product

$$\Psi_{AB} = \left(\sum_i a_i|\varphi_i\rangle\right) \otimes \left(\sum_j b_j|\psi_j\rangle\right) = \sum_{ij} a_i\, b_j|\varphi_i\rangle \otimes |\psi_j\rangle,$$

where $|\varphi_i\rangle \otimes |\psi_j\rangle$ is the basis of the state H_{AB} of the two particles (Plenio, 2002, p. 84). If the right side of the last equation always gives the middle side, the states of the form $|\varphi_i\rangle \otimes |\psi_j\rangle$ are

[25] This equation is in the Dirac notation; it is equivalent to the one presented in the footnote above (14).

called product states, but if the composite state cannot generate its substates, the states are called entangled states. Let us see such an example taken from Plenio's (2002) lecture notes: a state $|G\rangle$ can be written as a product

$$|G\rangle = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$= \alpha\gamma|0\rangle \otimes |0\rangle + \alpha\delta|0\rangle \otimes |1\rangle + \beta\gamma|1\rangle \otimes |0\rangle + \beta\delta|1\rangle \otimes |1\rangle;$$

in order for the equality to be true $\alpha\delta$ and $\beta\gamma$ would have to be zero, which leads to a contradiction since the equality cannot be true in the case the aforementioned constants are zero. Thus, we can conclude that the states $|0\rangle$ and $|1\rangle$ are entangled (p. 85).

While the mathematical formulation of entangled states can be seen as a consequence of Hilbert space as the state space, the notion of entanglement has a more profound implication that caused many physicists[26] and philosophers to deny it. This implication is non-locality that is the notion to physically influence something with no contact with it (Albert & Galchen, 2009). This notion clearly questions determinism[27] in science, which is the principle that there is a cause to everything that happens in nature (Reichenbach, 1965, p. 1). Indeed, determinism has been vital both to classical mechanics and to Einstein's relativity theory; that explained Einstein's resistance to entanglement that is epitomized in his famous quote "God does not play dice." This opposition is well expressed in the famous paper Einstein, Podolsky, and Rosen published in 1935 that showed quantum mechanics should be incomplete if it cannot give certain information

[26] One of prestigious physicists who did not believe that entanglement actually happens between particles is Albert Einstein who derogatorily called it "the spooky action at a distance." We shall talk about Einstein's opposition to entanglement later in the paper.
[27] The critique of determinism in science began before quantum mechanics since Boltzmann who formulated "Let us not forget that the principle of causality and the need for causality has been suggested to us exclusively by experiences with by experiences with macrocosmic phenomena, viz. the assumption that every individual occurrence be strictly causally determined has no longer any justification based on experience" (as cited in Reichenbach, 1965, p. 1).

about any state (p. 777); that would mean that the reason why entanglement seems nonlocal is because the wave function failed to give enough information, which again implies the hidden variables theory that there are hidden variables or parameters that preclude the scientist from knowing everything about the state of the system. This assumption had already been shown false by von Neumann in his theorem of impossibility, where he showed that the existence of hidden variables is impossible (Rosinger, 2004, p. 1). Let us see his logic; the subsequent material is from Rosinger (2004):

Neumann started with the assumption that hidden variables are necessary in giving an accurate picture of the state of the system; thus the Hilbert space needs to be added the set Λ of all the hidden variables, which is given by the cross product $H \times \Lambda$. The elements of the new space are of the form (ψ, λ) and called dispersion-free state. From this assumption, we can have these following properties:

1. $V(\psi, \lambda, \mathrm{A}) = \mathrm{E}_{\psi}(\mathrm{A})$, where V is a value function with real output, is$A$ an observable, and E is the expectation value operator.
2. $f\left(\mathrm{E}_{\psi}(\mathrm{A})\right) = \mathrm{E}_{\psi}\big(\mathrm{f}(\mathrm{A})\big)$, where f is a real-valued function with real arguments.

Neumann then looked for a general form for the function

$$\mathrm{E}_{\psi}: H \times \mho \to \mathbb{R}.$$

To do so, he set these three assumptions:

1. $E(\psi, \mathrm{I}) = 1$, where I is the identity operator and ψ an element of the Hilbert space.
2. $E(\psi, \alpha\mathrm{A} + \beta\mathrm{B}) = \alpha\mathrm{E}(\psi, \mathrm{A}) + \beta\mathrm{E}(\psi, \mathrm{B})$, where A and B are operators and α and β are real numbers.
3. $E(\psi, \mathrm{P}) \geq 0$, for all states of the Hilbert space. P is the projector operator.

Neumann then states this theorem[28]:

Theorem: The function E must have the form

$$E(\psi, \mathrm{A}) = tr\left(U_\psi A\right),$$

where U_ψ is a positive operator on H such that

$$tr\left(U_\psi\right) = 1.$$

If we consider the projector operator P, its expectation value can be only 0 or 1; if so, E should be a constant function since it is assumed to be continuous because of the hidden variables condition on it. If $E(\psi, \mathrm{P}) = 0$, $\langle\psi|P|\psi\rangle = 0$, which makes the trace of P zero that violates the above theorem. Also, if $E(\psi, \mathrm{P}) = 1$, $\langle\psi|P|\psi\rangle = 1$, which makes the trace of P different from 1 that also violates the above theorem. Thus, he concludes that the hidden variable theory claim does not hold since it leads to a contradiction (Rosinger, 2004, pp. 2-7).

Unfortunately, Neumann's proof happened to be wrong because the second of the three assumptions he made is not always true (Aczel, 2002, p. 141). This assumption is true only when the operators commute, where a simultaneous measurement would give a precise result (Rosinger, 2004, p. 7);

Definition: Two operators A and B commute if and only if

$$AB - BA = 0;$$

otherwise,

$$AB - BA = I,$$

where I is the identity operator (Byron Jr. & Fuller, 1992, p. 99).

[28] A proof to that theorem can be found in Rosinger's article.

If the operators do not commute, there is an uncertainty[29] in the measurement process; that is what Neumann failed to point out, and this is what the brilliant Irish particle physicist John Bell pointed out (Aczel, 2002, p. 141).

Bell's Inequalities

Indeed, in his paper, "On the Einstein Podolsky Rosen Paradox," Bell (1964) showed that there is a contradiction when implementing hidden variables in the measurement of quantum states, that which makes Einstein and others' argument false[30]. Let us then see how Bell came up with his proof[31]: Consideration is given to the Bohm-Aharonov interpretation of the EPR argument in which two photons are created in the singlet[32] spin[33] state moving in opposite direction and are measured by Stern-Gerlach magnets; if measurement on one component gives +1, the measurement on the other one must be -1[34] (Bell, 1964, p. 195). Let us assume that additional parameters λ are needed to give a complete result of the measurement on the particles; the values of the measurement on the particles are then

$$A(a, \lambda) = \pm 1; B(b, \lambda) = \pm 1,$$

where A, B are operators and a,b are unit vectors. Now, let us assume that the result A does not influence the result B; thus, the expectation value[35] of the product state is

[29] The famous Heisenberg uncertainty principle, which states that it is impossible to simultaneously measure both the position and the momentum of a particle, comes from this commutation rule; the position and momentum operators do not commute, which gives the relation $pq - qp = -i\hbar$, where p and q are respectively the position and the momentum operators (Byron, Jr. & Fuller, 1992, p. 99).

[30] While Eisntein's and other's reasoning was false (their proof was false), their paper showed an essential truth about quantum mechanics, which is that quantum mechanics and locality are incompatible: the more locality is considered, the less quantum mechanics is complete, but both cannot be true. This new way of interpreting the paper was developed by John S. Bell (Aczel, 2002, p. 143).

[31] This proof comes from his second paper, "On the Einstein Podolsky Rosen Paradox," published 1964.

[32] A singlet state is a "state having a total electron spin quantum number equal to zero" (IUPAC, 1997).

[33] Internal angular momentum

[34] The reason why the result of the second photon is determined by the other one is because the angular momentum of the system, which was originally zero because the photons come from the singlet state, is conserved. Thus, regardless of the distance between the particles, the determination of the spin of one photon simultaneously determines the spin of the other one (MathPage, n.d.).

[35] Remember that the expectation value of an observable A is $\langle A \rangle = \langle \varphi | A | \varphi \rangle$.

$$P(a,b) = \int d\lambda \rho(\lambda) A(a,\lambda) B(b,\lambda),$$

where $\rho(\lambda)$ is the probability distribution[36] of λ (Bell, 1964, p.). This expectation value should be equal to the quantum mechanical expectation value of the singlet state

$$\langle \sigma_1 . a \ \sigma_2 . b \rangle = a . b,$$

where σ_1, σ_2 are components of the spins of the photons; the purpose of the proof is to show that those expectation values cannot be equivalent, assuming that locality holds (Bell, 1964, p. 195). The proofs are taken from Bell's article "On the Einstein Podolsky Rosen Paradox."

Proof: Because ρ is a normalized probability distribution,

$$\int d\lambda \rho(\lambda) = 1,$$

and because $A(a,\lambda) = \pm 1; B(b,\lambda) = \pm 1$, P cannot be less than -1. It can reach -1 at a = b only if

$$A(a,\lambda) = -B(a,\lambda),$$

except at a set of points λ of zero probability. Assuming this, P can be rewritten

$$P(a,b) = -\int d\lambda \rho(\lambda) A(a,\lambda) A(b,\lambda).$$

It follows that c is another unit vector[37]

$$P(a,b) - P(a,c) = -\int d\lambda \rho(\lambda) [A(a,\lambda)A(b,\lambda) - A(a,\lambda)A(c,\lambda)]$$

---

[36] $\int_{-\infty}^{\infty} \lambda d\lambda = 1$

[37] One can easily see that $P(a,c) = -\int d\lambda \rho(\lambda) A(a,\lambda) A(c,\lambda)$. After adding the integrals, it is easy to factor by $d\lambda\rho(\lambda)$. One way one can obtain the second equality is to factor the right side by $-A(a,\lambda)$ that which cancels the negative sign outside of the integral sign

$$= \int d\lambda \rho(\lambda) A(a,\lambda) [A(c,\lambda) - A(b,\lambda)];$$

the expression within the brackets can be factored by $A(b,\lambda)$ since $A(b,\lambda)$ can be equal to 1, which is

$$= \int d\lambda \rho(\lambda) A(a,\lambda) A(b,\lambda) [A(c,\lambda) - 1];$$

by multiplying the first term in the brackets by $A(b,\lambda)$, expression becomes

$$= \int d\lambda \rho(\lambda) A(a,\lambda) A(b,\lambda) [A(b,\lambda)A(c,\lambda) - 1].$$ *Author's Note.*

$$= \int d\lambda\rho(\lambda)A(a,\lambda)A(b,\lambda)[A(b,\lambda)A(c,\lambda) - 1]\,;$$

using that $A(a,\lambda) = \pm 1; B(b,\lambda) = \pm 1$, we then have

$$|P(a,b) - P(a,c)| \leq \int d\lambda\rho(\lambda)[1 - A(b,\lambda)A(c,\lambda)].$$

The second term on the right is[38] $P(b,c)$, so

$$1 + P(b,c) \geq |P(a,b) - P(a,c)|.$$

Unless P is constant, the right hand side is in general of order $|b - c|$ for small $|b - c|$. Thus, $P(b,c)$ cannot be stationary at the minimum value -1 at b = c and cannot equal the quantum mechanical expectation value.

It can also be shown that the quantum mechanical correlation cannot be arbitrarily closely approximated by P(a, b).

Proof: Let us consider the functions

$$\bar{P}(a,b)\text{and}\,\overline{-a.b},$$

where the bar denotes independent averaging of P(a',b') and –a'.b' over vectors a' and b' within specified angles of a and b. Suppose that for all a and b the difference is bounded by ϵ, we then have

$$|\bar{P}(a,b) + a.b| \leq \epsilon \ \ (1).$$

Then it will be shown that ϵ cannot be arbitrarily small. Suppose that for all a and b

$$\left|\overline{a.b} - a.b\right| \leq \delta \ \ (2)$$

Then from (1),

$$|\bar{P}(a,b) + a.b| \leq \epsilon + \delta \ \ (3).$$

From the correlation formula we have had earlier,

[38] To get 1+ P(b, c), on just needs to expand the integrand in the above line.

$$\bar{P}(a,b) = \int d\lambda\rho(\lambda)\bar{A}\,(a,\lambda)\bar{B}(b,\lambda) \;\; (4),$$

where

$$|\bar{A}(a,\lambda)| \leq 1;\; |\bar{B}(b,\lambda)| \leq 1 \;\;\; (5).$$

From (4) and (5), with a = b,

$$d\lambda\rho(\lambda)[\bar{A}(a,\lambda)\bar{B}(b,\lambda) + 1] \leq \epsilon + \delta \;\;\; (6)\,.$$

From (4),

$$\bar{P}(a,b) - \bar{P}(a,c) = \int d\lambda\rho(\lambda)[\bar{A}\,(a,\lambda)\bar{B}(b,\lambda) - \bar{A}(a,\lambda)\bar{B}(c,\lambda)]$$

$$= \int d\lambda\rho(\lambda)\bar{A}\,(a,\lambda)\bar{B}(b,\lambda)[1 + \bar{A}(a,\lambda)\bar{B}(c,\lambda)]$$

$$- \int d\lambda\rho(\lambda)\bar{A}\,(a,\lambda)\bar{B}(c,\lambda)[1 + \bar{A}(b,\lambda)\bar{B}(b,\lambda)]$$

Using (5) then

$$|\bar{P}(a,b) - \bar{P}(a,c)| \leq \int d\lambda \propto (\lambda)\,[1 + \bar{A}(a,\lambda)\bar{B}(c,\lambda)] \;\; + \int d\lambda\rho(\lambda)\,[1 + \bar{A}(b,\lambda)\bar{B}(b,\lambda)].$$

Then using (4) and (6),

$$|\bar{P}(a,b) - \bar{P}(a,c)| \leq 1 + \bar{P}(b,c) + \epsilon + \delta.$$

Finally,

$$|a.c - a.b| - 2(\epsilon + \delta) \leq 1 - b.c + 2(\epsilon + \delta)$$

or

$$4(\epsilon + \delta) \geq |a.c - a.b| + b.c - 1.$$

Take for example $a.c = 0, a.b = b.c = \frac{1}{\sqrt{2}}$; then,

$$4(\epsilon + \delta) \geq \sqrt{2} - 1.$$

Therefore, for small finite δ, ϵ cannot be arbitrarily small[39]. Thus, the quantum mechanical expectation value cannot be represented, either accurately or arbitrarily close, in the form of P (a, b).

If systems of dimension higher than two are considered, a subspace of two dimensions can be considered and the same proof can be performed. That would show that, for at least one quantum mechanical system, the statistical predictions of quantum contradict hidden variables hypothesis (Bell, 1964, p.199).

Quantum Entanglement Experiments: Violation of Bell's Inequalities

CSHS proposed experiment. Although Bell presented a rigorous proof to show that the hidden variables theory does not hold, the proof had more of a mathematical significance than a physical relevance; the ideal way to ascertain that Bell was right was to put his proof under physical test to see if it holds true. Indeed, this is exactly what Clauser, Horne, Shimony, and Holt (1969) had in mind when they published their article, "Proposed Experiment to Test Local Hidden-Variable Theories," in which they derived a more general form of Bell's inequality for practical reason and proposed an experiment[40] based on Kocher and Commins'[41] experiment and Bohm's *Gedankenexperiment*[42] that Bell had used for his proof (Clauser, Horn, Shimony, & Holt, 1969, p. 880).

Figure 4.

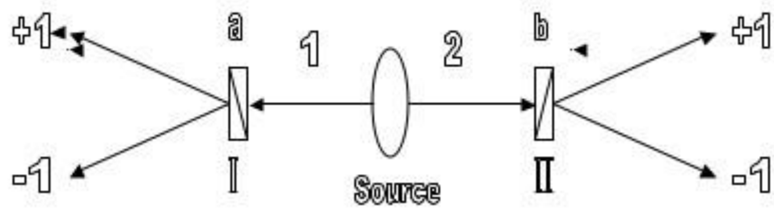
[39] Remember that the initial goal was to see if the two expectations differ only by a small positive infinitesimal number; if so, it could have been deduced that those expectation values are very close to each other. This would be possible if both δ and ϵ can be very small. However, it is shown that this is not possible, meaning that the two expectation values cannot get close to each other. *Author's Note*.

[40] In 1972, Freedman and Clauser performed the experiment that first confirmed that quantum mechanics is non-local (Aczel, 2001, p. 172).

[41] Kocher and Commins' experiment used the setting in Bohm's Gedankenexperiment to measure the polarization correlation of the pairs of photons created by the cascade of calcium. It was then shown that the Bell's inequalities are violated. However, according to Clauser et al., the data were not sufficient because of the poor efficiency of the polarizers and because of the orientations considered, which were zero and ninety degrees (Clauser, Horn, Shimony, & Holt, 1969, p. 882).

[42] "Thought experiment" in German

Bohm's *Gedankenexperiment*



Note. This table is taken from the article "Bell Theorem" from the *Stanford Encyclopedia of Philosophy* website.

Let us see how they derive the general form of Bell inequalities and what they propose as experiment[43]: The correlation function is defined by this formula that is the same at Bell's

$$P(a,b) \equiv \int_{\Gamma} A(a,\lambda)B(b,\lambda)\rho(\lambda)\,d\lambda,$$

where Γ is the total λ space, we then have

$$|P(a,b) - P(a,c)| \leq \int_{\Gamma} |A(a,\lambda)B(b,\lambda) - A(a,\lambda)B(c,\lambda)|\rho(\lambda)\,d\lambda$$

$$= \int_{\Gamma} |A(a,\lambda)B(b,\lambda)|[1 - B(b,\lambda)B(c,\lambda)]\rho(\lambda)\,d\lambda$$

$$= \int_{\Gamma} [1 - B(b,\lambda)B(c,\lambda)]\rho(\lambda)\,d\lambda$$

$$= 1 - \int_{\Gamma} B(b,\lambda)B(c,\lambda)\rho(\lambda)\,d\lambda.$$

Suppose that for some b' and b, $P(b',b) = 1 - \delta$ where $0 \leq \delta \leq 1$. Here Bell's experimentally unrealistic restriction that for some pair of parameters b' and b there is perfect correlation $(\delta = 0)$. Dividing Γ into two regions Γ_+ and Γ_- such that $\Gamma_\pm = \{\lambda | A(b',\lambda) = \pm B(b,\lambda)\}$, we have $\int_{\Gamma_-} \rho(\lambda)\,d\lambda = \frac{1}{2}\delta$. Hence,

[43] The notation used by Clauser et al. for correlation is slightly different from what Bell used.

$$\int_{\Gamma} B(b,\lambda)B(c,\lambda)\rho(\lambda)\,d\lambda = \int_{\Gamma} A(b',\lambda)B(c,\lambda)\rho(\lambda)\,d\lambda - 2\int_{\Gamma_-} A(b',\lambda)B(c,\lambda)\rho(\lambda)\,d\lambda$$

$$\geq P(b',c) - 2\int_{\Gamma_-} |A(b',\lambda)B(c,\lambda)|\rho(\lambda)\,d\lambda = P(b',c) - \delta;$$

Therefore, the new inequality[44] is

$$|P(a,b) - P(a,c)| \leq 2 - P(b',b) - P(b',c).$$

In the proposed experiment, P (a, b) depends only on the parameter difference b-a; let us then define these new variables as such: if

$$\alpha \equiv b - a, \beta \equiv c - b, \text{and } \gamma \equiv b - b',$$

then

$$|P(\alpha) - P(\alpha+\beta)| \leq 2 - P(\gamma) - P(\beta+\gamma).$$

Now that a more general form is obtained from Bell's inequalities, experimental quantities, called coincidence rates, are needed to calculate the correlation[45] between the two photons based on their emergence or non-emergence from the linear polarization filters (Clauser,…, 1969, p. 881). We shall then see how the inequalities are given with respect to coincidence rates: if we assume that the probability of the joint detection of the pair of photons emerging from I and II (figure 3) is independent of a and b and if the flux into I and II is a constant independent of a and b, the rate of coincidence detection R (a, b) will be proportional to $w[A(a)_+, B(b)_+]$, where $w[A(a)_\pm, B(b)_\pm]$ is the probability that $A(a) = \pm 1$ and $B(b) = \pm 1$. Putting

$R_0 = R(\infty,\infty), R_1(a,\infty), and\ R_2(\infty,b),$

$$P(a,b) = w[A(a)_+, B(b)_+] - w[A(a)_+, B(b)_-]w[A(a)_-, B(b)_+] - w[A(a)_-, B(b)_-]$$

and

[44] Another well-known version of this inequality is $-2 \leq S \leq 2$, where $S = P(a,b) - P(a,b') + P(a',b) + P(a',b')$ (Aspect, Grangier, & Roger, 1982, p. 91).

[45] P (a,b) is the correlation; in the article, it is called emergence correlation function.

$$w[A(a)_+, B(\infty)_+] = w[A(a)_+, B(b)_+] + w[A(a)_+, B(b)_-],$$

we have

$$P(a,b) = \frac{4R(a,b)}{R_0} - \frac{2R_1(a)}{R_0} - \frac{2R_2(b)}{R_0} + 1.$$

So, the absolute value of the difference of the correlation presented above can be expressed with respect of coincidence rates. If $R_1(a)$ and $R_2(b)$ are found experimentally to be constant R_1 and R_2, the result is

$$|R(a,b) - R(a,c)| + R(b',b) + R(b',c) - R_1 - R_2 \leq 0.$$

When P(a, b) = P(a-b), we then have

$$|R(\alpha) - R(\alpha+\beta)| + R(\gamma) + R(\beta+\gamma) - R_1 - R_2 \leq 0.$$

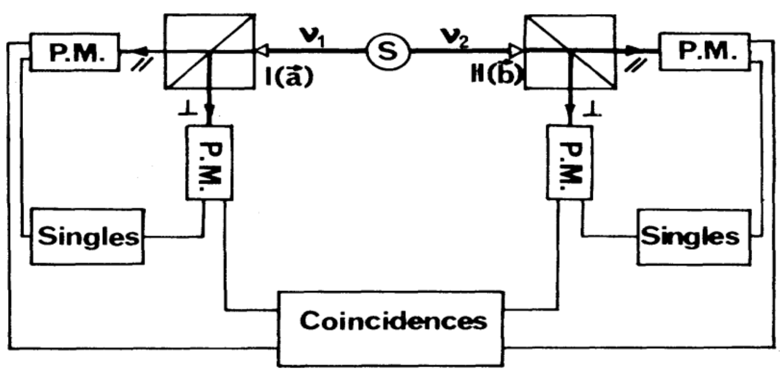
In the proposed experiment, Kocher and Commins' experiment is modified to consider more orientations of the polarizers. Measurement will be taken with one polarizer removed and then the other removed. With efficient polarizers, preferably polarizers with antireflection coatings, it is predicted that the greatest violation of the extended Bell's inequalities happens at $\alpha = 22.5^\circ, \beta = 45^\circ,$ and $\gamma = 157.5^\circ$ for a 0-1-0 cascade and at $\alpha = 67.5^\circ, \beta = 135^\circ,$ and $\gamma = 112.5^\circ$ for a 0-1-1 cascade; one can see that the four angles in the inequality above are reduced to just 22.5 and 67.5 degrees, which correspond to the orientations of the polarizers (Clauser, Horne, Shimony, & Holt, 1969, p. 882-883).

Experiment with two-channel analyzers. Although the proposed experiment by Clauser et al. and Clauser-Freedman experiment were considered as the proof that quantum mechanics won the battle against hidden variables theory, they still had some weaknesses that were criticized by other physicists: the major skepticism came from the problem of unobserved photons (Aczel, 2001, p. 172). Among those skeptics were Aspect, Grangier, and Roger (1982) who pointed out that experiments that had been performed did not accurately prove that quantum mechanics violate Bell's inequalities because the pairs of photons could not be measured simultaneously[46] (p. 91). Therefore, they designed and performed a new experiment where those "loopholes" are considered. Let us see the procedures and result of their experiment, which was published in the *Physical Review Letters* under the title of "Experimental Realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A New Violation of Bell's Inequalities":

Procedures. "Dichotomic polarization" measurements are performed simultaneously with two-channel polarizers followed by two photomultipliers (figure 5). In a single run, the four coincidence rates $R_{\pm\pm}(a,b)$ can easily be measured by using a fourfold coincidence technique. The new correlation function is then

$$P(a,b) = \frac{R_{++}(\vec{a},\vec{b}) + R_{--}(\vec{a},\vec{b}) - R_{\pm}(\vec{a},\vec{b}) - R_{\mp}(\vec{a},\vec{b})}{R_{++}(a,b) + R_{--}(a,b) - R_{\pm}(a,b) - R_{\mp}(a,b)}.$$

Figure 5.
Experimental Setup



Note: Two polarizers I and II, in orientations a and b, performed true dichotomic measurement of linear polarization on photons v1 and v2. Each polarizer is rotatable around the axis of incident beam. The counting electronics monitors the singles and the coincidences. This graph is taken from the article.

The same measurement is repeated for the three other choices of orientations. The source in this experiment is a $(J = 0) \rightarrow (J = 1) \rightarrow (J = 0)$ cascade in calcium-40 with use of two single-mode lasers.

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[46] Recall that, in the proposed experiment, it is suggested that one polarizer be removed and then the other. Also, Aspect, Grangier, and Roger replaced the single-channel polarizer by a two-channel polarizer for simultaneous measurement.

Result. When five runs are performed at each of the orientations where greatest violation happens, the average value of S is

$$S_{expt} = 2.697 \pm 0.015,$$

which is in a less than 1% agreement[47] with quantum prediction of S

$$S_{QM} = 2.70 \pm 0.05.$$

Again, this experiment reinforced the notion of entanglement between the pairs of electrons. Later, another experiment[48], under the title "Experimental Test of Bell's Inequalities Using Time-Varying Analyzers," was performed by Aspect, Dalibard, and Roger (1982) to include variable analyzers whose absence in the first experiment was considered as a loophole.

After these experiments, many other experiments have been performed that are in favor of quantum mechanics. In 2008, Salart, Baas, van Houwelingen, Gisin, and Zbinder (2008) published their Bell's test experiment where the inequalities are still violated with pairs of particles 18 Km apart from each other (p. 1).

After we have presented the historical background of quantum mechanics, the mathematical formalism of the theory, the theoretical explanation of quantum entanglement, and its experimental validity, one may conclude that quantum entanglement is a real phenomenon and that its spookiness, alluded by Einstein, merely is a manifestation that classical physics, which has shaped humans' intuition for years, cannot describe all the intricacies of nature.

[47] This is the strongest violation of the inequalities.

[48] For this experiment, $S_{expt} = 0.101 \pm 0.020$ and $S_{QM} = 0.112$, which violates the inequality $S \leq 0$ by 5 standard deviation (Aspect, Dalibard, & Roger, 1982, p. 1807).

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