Gabriel's Horn:

An Understanding of a Solid with Finite Volume and Infinite Surface Area[1]

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Abstract

The Gabriel's Horn, which has finite volume and infinite surface area, is not an inconsistency in mathematics as many people think. Although it is inconceivable with a Euclid-based logic, it is very logical with modern mathematics. The infinite surface area of the solid is very reasonable due to the exponential rate of change of hyperbolic surfaces when it tends to infinity. Also, its finite volume can well be compacted into a finite spherical cap by projective and conformal transformations. Furthermore, the volume $V = \pi \int_1^\infty (1/x^2)\, dx$ can be easily calculated after testing for convergence of the improper integral.

[1] I thank Professors I. Danastor and J. Campbell and the editors for their useful insight, and I greatly appreciate Robert Slattery's help in proofreading this paper.

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Gabriel's Horn:[2] An Understanding of a Solid with Finite Volume and Infinite Surface Area

Is it possible to cover the surface of earth with a pea[3]? This question may seem bizarre because it seems not to make any sense: a pea of a few cubic centimeters obviously cannot contain Earth's$(4/3)\pi \times 6400000^3 m^3$. However, when carefully thinking about the question considering its inferences, one will discover that, although the literal pea may not do the work, some other solids may. A perfect example is the Gabriel's Horn, which has finite volume and infinite surface area: it can cover millions of Earths. This solid has been the center of controversy since its discovery by Evangelista Torricelli in 1641 because it is "logically" impossible to have that kind of solid based on what intuition dictates (Mancosu & Vailati, 1991, p. 50). This discovery not only violently struck the mathematical community at that time because it violated secular ideas about geometric figures, but it also shook the philosophical assumption about infinity (Mancosu & Vailati, 1991, p.50). However, what is surprising is that the ambiguity evolved from the counterintuitive property of this solid is logically consistent with modern mathematical theories. In order to understand why it contradicts traditional ideas and how it works perfectly with more recent ones, a development of the ancient Euclidean geometry is necessary along with the recent non-Euclidean geometry; also, in a deep study of this solid, two analytical methods will be presented to illustrate that the volume is finite.

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[2]The name comes from the archangel Gabriel who, in the Bible, used a horn to announce good news, the birth of Jesus, or bad news, Armageddon (Fleron, 1999, p.35). This solid is also known as the Torricelli's trumpet.

[3]A version of a famous mathematical paradox, the Banach-Tarski paradox, is that a pea can cover the surface of the sun. In this paradox, the surface of the pea is not considered to be infinite; however, the points inside the pea can be grouped into pieces and those pieces, when rotated in a specific way and reassembled, can cover all the points in the sun. One needs to know that pea and sun are all mathematical objects assumed to be composed of points. There is a more formal definition of this paradox is given by the theorem, "If A and B are any subsets of the spherical surface $S^2 = \{(x, y, z): x^2 + y^2 + z^2\}$, both having non-empty interiors, then there exists two finite partitions into disjoint pieces $A = A_1 U A_2 U \dots . U A_n$ and $B = B_1 U B_2 U \dots . U B_n$, such that A_1 is congruent to B_1 by rotation $\rho_i$of the sphere for $i = 1, \dots, n$" (Mycielski, 1987, p. 698). This theorem can be proved by the Zermelo's axiom of choice that states that if considered a set of disjoint non-empty subsets, there exists at least one set that has exactly one element in common with each subset. Like the Gabriel's horn, this axiom is controversial, and many mathematicians refuse to use it while many praise it.

Euclidean Geometry

Before beginning developing the whole body of mathematical knowledge built by Euclid (300), a history of this powerful man would be necessary. Euclid (or Ευκλειδης in Greek) lived in a period when Alexandria was founded and where science, art, and several other forms of knowledge flourished. Indeed, 331 B.C. knew significant changes; for example, this is at the same epoch that the lighthouse at Faros was built; such a monument was considered one of the world's Seven Wonders. In this period of extensive progress, Euclid has greatly contributed to mathematics by inventing a new way of thinking (Holme, 2002, p. 67). The fundamental idea on which Euclid's work is based is clearly defined by "The Hypothetical-Deductive Method," which says that:

> All known geometric facts or theorems should be deduced by agreed upon logical rules of reasoning from a set of initial, self evident truths, called *postulates*. The postulates should be such that every informed person would agree on their validity, to the extent that they did not require proof. The set of postulates should be kept as small as possible, thus one should endeavor to construct proofs of assertions which, even though self evident, could be deduced from other even more fundamental self evident ones. (Holme, 2002, p. 68)

This is how Euclid constructs his geometry, where he defines more complex figures or properties of these figures by using the fundamental concepts, such as points, lines, or planes.

Indeed, Euclid based his geometry on five axioms, which are self-evident principles that are accepted as true without any proof, and five postulates, which are considered less obvious than axioms. Those axioms and postulates are well developed in his *The Elements* composed of thirteen books, which are divided into three main categories: plane geometry, Books I-IV;

arithmetic, Books VII-X; and solid geometry, Books XI-XII (Artmann, 1999, p. 3). The focus will be on Book I because it develops the basis of Euclidean geometry, where the notions of points, lines, and angles are defined. Artmann's *Euclid-The Creation of Mathematics* is the book that will be used to develop the definitions, postulates, propositions in *The Elements*. Some of the definitions considered are:

1. A point is that which has no part.
2. A line is breadthless length.
3. The extremities of a line are points.
4. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a strange line.
5. And when the lines containing the angle are straight, the angle is called rectilinear.
6. A surface is that which has length and breadth only.
7. A plane surface is that which lies evenly with the straight lines on itself.
8. A solid is that which has length, breadth, and depth.
9. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction (p.18)

One needs to understand that a point and a line are all abstract, that is, one needs not expect to find exact real representation of an infinite breadthless line or a zero-dimension point with no mass; also, Euclidean geometry is the geometry on a flat plane. In the book, there are twenty-two definitions, but only the definitions that are relevant to our objective are presented.

Later, Euclid comes with his five famous postulates:

1. Let it be postulated to draw a straight line in a straight line,
2. To produce a limited straight line in a straight line,

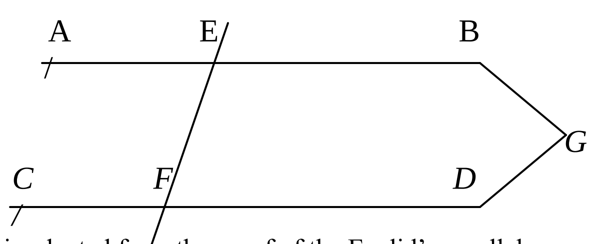
3. To describe a circle with any center and distance,
4. That all right angles are equal to each other.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles (Artmann, 1999, p. 19).

The fifth postulate is very important because it holds the key for the development of other types of geometry that we shall develop later. As a result, proposition I.27 in *The Elements* on parallel lines and its proof are presented:

Prop. I. 27: *If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.*

Proof: For let the straight line EF falling on two straight lines AB, CD make the alternate angles AEF, EFD equal to one another (Fig. 1); AB is parallel to CD.

Figure 1



Source: This figure is adapted from the proof of the Euclid's parallel proposition by B. Artmann, 1999, *Euclid-The Creation of Mathematics* p. 32.

For, if not, AB, CD when produced will meet either in the direction of B, D or towards A, C. Let them be produced and meet, in the direction of B, D at G. Then, in the triangle GEF, the exterior angle AEF is equal to the interior and opposite angle EFG: which is impossible. Therefore AB, CD when produced will not meet in the direction of B, D. Similarly it can be proved that neither will they meet towards A, C.

But Straight lines which do not meet in either direction are parallel; therefore AB is parallel to CD [Proved in Prop. I. 16] (Artmann, 1999, p. 31-32).

In addition to definitions and postulates, there are axioms that deal with more than geometric figures; Euclid names them "common notions," which are:

1. Things equal to the same thing are also equal to one another.
2. If equals are added to equals the wholes are equal.
3. If equal are subtracted from equals the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part (Artmann, 1999, p. 19)

One needs to notice that a Euclidean surface is composed of lines that are straight and that most Euclidean solids are also composed of straight lines, which do not always constitute all solids, particularly solids that are formed by curved lines.

Euclid's geometry had been considered the ultimate truth in mathematics for years, and the *Elements* has been one of the most important scientific books for centuries. To have an idea of how well this book was respected, it was one of the first mathematics books to be printed after printing presses had been invented; also, even Abraham Lincoln chose it as his favorite book to have a peaceful sleep (O'Shea, 2007, p. 55). Another striking, if not shocking, example of the value of Euclid in some societies is the interpretation or the use of his book to fulfill some needs that have almost nothing to do with science: in the United States, it was believed that the book had a special luck with women; consequently, students at Mount Holyoke College, an all women college, would be required to have and memorize Simson's or Playfair's Euclid, the fifth postulate (O'Shea, 2007, p.55). Then, one needs to see that the Euclidean mathematical system went beyond the boundary of science; it had even shaped how many people saw the world in which they live and helped them realize some fantasies that they could not by themselves. One can now understand why challenging the base of Euclidean geometry would cause great tumult not only in the mathematical community but also in the overall society.

Non-Euclidean Geometry

Indeed, controversy arose when a group of young mathematicians revealed that Euclid's fifth postulate is not always true or not true at all. This postulate had long been considered to have something that is not quite true. Greeks writers had tried to give a more rational explanation of it but failed (Gray, 2004, p. 20). Also, O'Shea (2007) pointed out that the length of the fifth postulate compared to the four other ones might suggest that even Euclid knew about the ambiguity of that postulate; therefore, he took more time to explain it (p. 57). One needs to know that the fifth postulate is central to Euclid's *Elements*; without it, a great part of his logical body of mathematics would not have existed. For example, the parallel postulate is used to demonstrate the Theorem of Pythagoras, which states that the square of the greater side in a right triangle is equal to the sum of the squares of the two other sides; also, the theorem that the sum of the angles in a triangle is exactly two right angles and the construction of similar triangles depend exclusively on the fifth postulate (Gray, 2004, p. 19-20). Those three young mathematicians, Johann Carl Friedrich Gauss (1777-1855), Nikolai Ivanovich Lobachevsky (1792-1856), and Janos Bolyai (1802-60) brought a significant contribution to challenge the fifth postulate (O'Shea, 2007, p. 62). Those brilliant men discovered that other geometries were consistent without the fifth postulate; those geometries, under a generic name of non-Euclidean geometry, are usually called elliptic geometry, spherical geometry, and hyperbolic geometry. Focus will be on hyperbolic geometry since part of it will be used to explain why the solid given by the revolution of a hyperbola has finite volume and infinite surface area.

Furthermore, Riemann, a prominent Gauss's student, revolutionized the area of geometry by radically considering surfaces as a space in themselves (Maor, 2007, p. 174). This critical change resulted in studying other spaces that are not Euclidean, which are called curved space. Since those spaces are not Euclidean, their properties are often paradoxical, compared to Euclidean properties that often are easily understood. Some notions of this new version of geometry will be used to evaluate the surface area of the Horn.

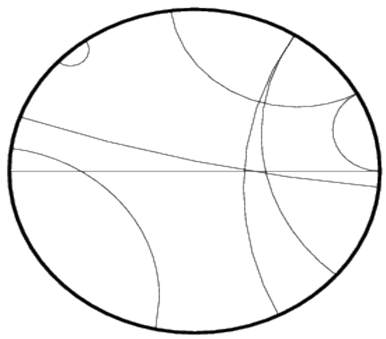
Hyperbolic Geometry

An easy definition of hyperbolic geometry is that it is a non-Euclidean geometry in which the fifth postulate does not hold; the parallel postulate for the hyperbolic plane is "Given a line and a point outside it. Then there are at least two lines through the point which do not meet the line" (Holme, 2002, p. 195). Before a graphical representation of this postulate is given, more basic notions need to be defined: Points on a hyperbolic plane are the same as points on a Euclidean plane, but hyperbolic lines[4] are different from Euclidean lines in that they curve. Also, those properties that are true with Euclidean lines may be false with hyperbolic lines: if two lines are parallel to a third, these lines are parallel to each other; if two lines are parallel, the distance between them will be constant everywhere; and infinite lines do not have boundary (Castellanos, 2007, para. 2). In order to better understand those counterintuitive properties of lines, Euclidean representation of non-Euclidean planes, called models, are presented. The most used ones are the Klein model, the hyperboloid model, the Poincaré disc model, and the Poincaré half-plane model. For the purpose of this paper, the two last models are presented.

[4]Although hyperbolic lines are curved on a Euclidean place, they are "straight" on a hyperbolic surface. The notion of straightness does not exist only in Euclidean geometry.

In the disc model, points in the non-Euclidean space are inside a circle called the Boundary Circle; points on the boundary are not part of the space and do not have any functions. Non-Euclidean straight lines are either Euclidean straight lines perpendicular to the boundary circle or arc of circles perpendicular to the boundary circle (Gray, 2004, p.93).

Figure 2
Poincaré Disk Model

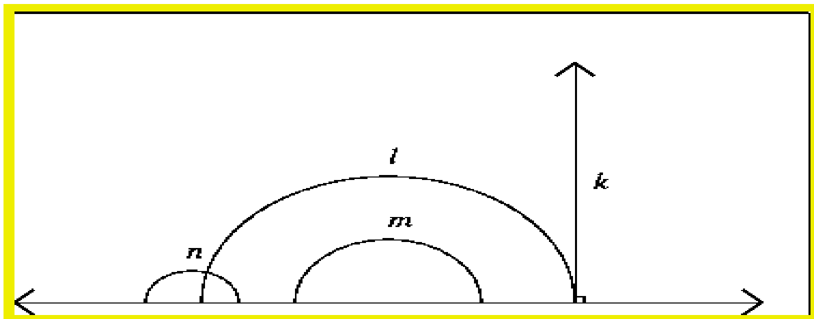


Note. The Disk Model is taken from "Poincaré Hyperbolic Disk" by E. W. Weisstein, 2008a from http://mathworld.wolfram.com/PoincareHyperbolicDisk.html.

In the circle, lines are infinite. One can wonder how those lines can be infinite. One property in hyperbolic geometry is that things get smaller when they get closer to the Boundary Circle; this means that the lines will never touch the circle even if they are "infinitely" long. Also, there is not a constant distance between two parallel lines (Castellanos, 2007, para. 2).

On the other hand, lines are represented by semicircles with their centers on the x-axis in the Poincaré Half-Plane model. Those lines obey all Euclidean properties of line except the parallel postulate.

Figure 3
Poincaré Half-Plane Model



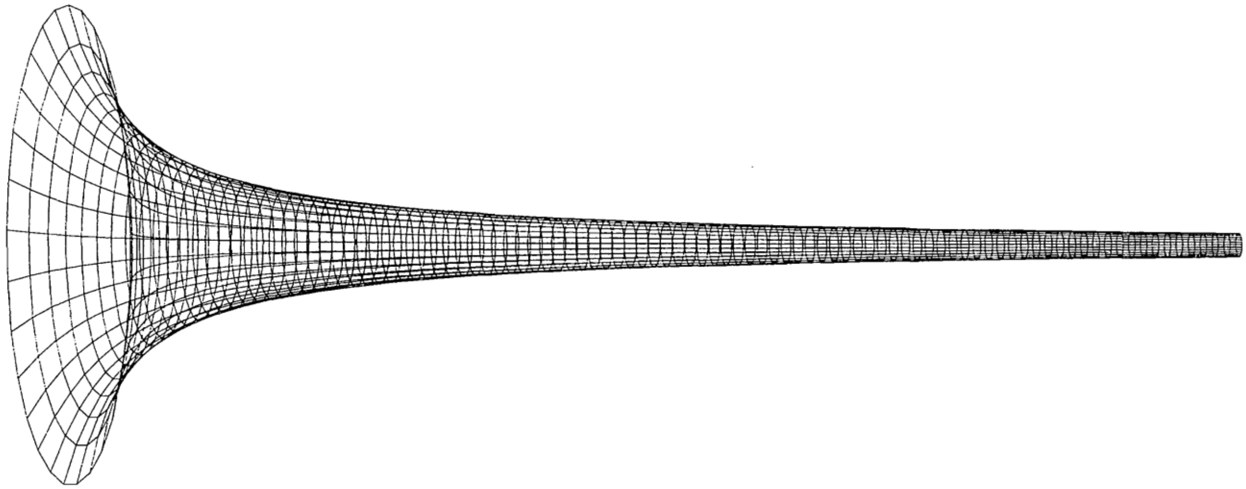
Note. This figure is taken from “The Upper Half Plane” from http://www.geom.uiuc.edu/~crobles/hyperbolic/hypr/modl/uhp/.

In this plane, line *n* is parallel to *m* that is parallel to *l*; however, *n* is not parallel to *l,* which contradicts the Euclidean rule which states that if two lines are parallel to a third, those lines are parallel to each other.

Previously, some concepts of non-Euclidean geometry have been developed to fully explain the counterintuitive properties of the Gabriel’s Horn. Now, the method of construction of the solid is presented along with the properties that violate some Euclidean rules.

Gabriel’s Horn is formed by revolving the rectangular hyperbola given by the function $1/x$ around the x-axis with $x \geq 1$. A more formal definition of hyperbolic geometry is needed: “Hyperbolic geometry is a non-Euclidean geometry that has a constant sectional curvature -1” (Weisstein, 2008b, para. 1). A sectional curvature calculates the rate of change of the geodesic deviation, and a hyperbola is the geodesic on Gabriel’s Horn. Thus, it will be shown that the curvature of the hyperbola is -1 to conclude that the surface of the solid does not constitute a Euclidean plane, which has a curvature 0.

Figure 4
Gabriel's Horn



Note. This figure is taken from the article "Gabriel's Wedding Cake" by J. F. Fleron, 1999, *The College Mathematical Journal*, *30*, pp. 35-38.

The parametric equations of the branch of a rectangular hyperbola is

$$x = \cosh t$$
$$y = \sinh t \; .$$

The formula of a curvature *k* of a plane given parametrically is given by the formula

$$k = F[x, y] = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{\frac{3}{2}}} \; .$$

When replacing the values of x and y of the parametric equations of the hyperbola in *k*,

$$k = -\frac{1}{[\cosh(2t)]^{\frac{3}{2}}}.$$

To find t in *k*, x and y are replaced by 1 in the parametric equations because y = 1/x. x is easily found by considering one of the parametric equations and replacing it by their exponential equivalent; thus,

$$\cosh t \quad = 1$$

$$\frac{e^t + e^{-t}}{2} = 1$$

$$e^{2t} - 2e^{t} + 1 = 0$$

$$t = 0\,.$$

And by replacing t by 0 in *k*,

$$k = -\frac{1}{[\cosh(0)]^{\frac{3}{2}}} = -\frac{1}{1} = -1\,.$$

Since the curvature of the hyperbola is -1 and hyperbolas constitute the surface of the Gabriel's Horn, we easily can understand that the surface of the solid constitutes a non-Euclidean plane of curvature -1 as Euclidean lines with a curvature 0 constitute a Euclidean plane of curvature 0.

Now that it is established that Gabriel's Horn is not Euclidean, it will geometrically be demonstrated how the surface area can be infinite and the volume finite. A surface of an object is a two-dimensional manifold whose representation is possible on a sheet of paper. It is two-dimensional because the plane is generated by two independent vectors (O'Shea, 2007, p. 22). Furthermore, a surface area is the area of the surface; it is the measure of how much surface is exposed (Weisstein, 2008c, para. 1). In order to show how the surface of the solid is infinite, the surface of the solid is divided into non-Euclidean non-overlapping triangles; Euclidean triangles would not work because they do not have the curvature of the solid in question. Besides, the sum of the angles in a hyperbolic triangle is less 180 degrees; this sum can be calculated by the formula

$$T = \iint k dA$$

where *k* represents the curvature of the surface (Graustein, 1962, p.188). This formula will not be demonstrated because such a proof is not relevant to the subject. Thus, the area of the solid is the sum of the area of the triangles.

Now, let us analytically show how the area can be infinite. The parametric equations for the surface of Gabriel Horn when revolved around the z-axis are

$$z = u$$

$$x = f(u)\cos(v)$$

$$y = f(u)\sin(v).$$

Definition. *Let S be a smooth parametric surface*

$$\boldsymbol{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

defined over an open region D in the uv-plane. If each point on the surface S corresponds to exactly one point in the domain D, then the surface area of S is given by

$$Surface\ area = \iint dS = \iint \|\mathbf{r}_u \times \mathbf{r}_v\| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\ and\ \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\ \text{(Larson et al, 2006, p. 1102).}$$

The parametric surface of the Horn is then

$$\mathbf{r}(u,v) = f(u)\cos(v)\,\mathbf{i} + f(u)\sin(v)\,\mathbf{j} + u\mathbf{k}.$$

Now,

$$\mathbf{r}_u = f'(u)\cos(v)\,\mathbf{i} + f'(u)\sin(v)\,\mathbf{j} + \mathbf{k}\ \text{and}\ \mathbf{r}_v = -f(u)\sin(v)\,\mathbf{i} + f(u)\cos(v)\,\mathbf{j}.$$

So, the cross product of those vectors is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \boldsymbol{j} & \mathbf{k} \\ f'(u)\cos(v) & f'(u)\sin(v) & 1 \\ -f(u)\sin(v) & f(u)\cos(v) & 0 \end{vmatrix}$$

$$= -f(u)\cos(v)\,\mathbf{i} + f(u)\sin(v)\,\mathbf{j} + f(u)f'(u)\mathbf{k}.$$

Then,

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{[f(u)]^2 + [f(u)f'(u)]^2}$$

$$= f(u)\sqrt{1 + [f'(u)]^2}.$$

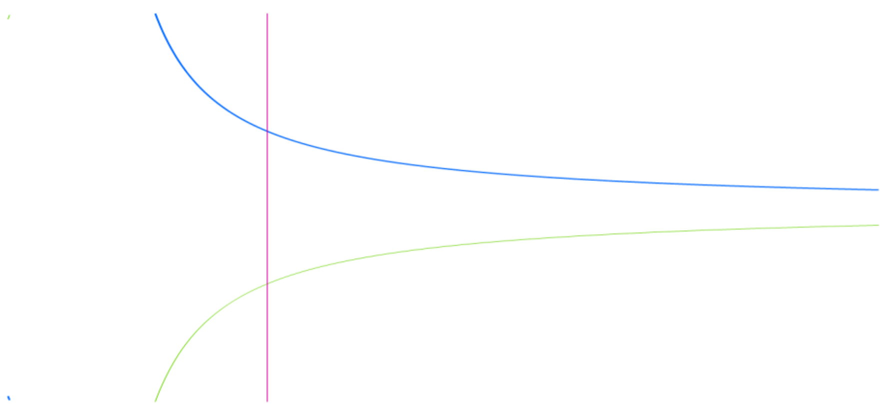
Thus, the surface area A of the solid is

$$A = \int_0^{2\pi} \int_1^{\infty} f(u)\sqrt{1 + [f'(u)]^2}\, dudv = \infty;$$

the reason why the area is infinite is because the interior integral evaluates the length of the geodesic line on the Horn, which is infinite. Revolving around the z-axis still makes it infinite since size of objects is invariant under rotation.

Another reason why this solid with infinite length could never exist based on Euclidean assumptions is that when each longitudinal lines diametrically opposite to each other form an angle less than 90 degrees to a vertical line crossing them, they never touch; according to the Euclid's fifth postulate, those lines would have to touch at a point, which would make the solid finite both in volume and surface (figure 5).

Figure 5
Longitudinal Cross-Section of the Horn



Since the solid is infinite, the surface area of the solid is also infinite. The infinite size of the area of the solid is less interesting since one can make the "rational" deduction that an infinite solid should always have an infinite surface area, although such a solid may be only in the realm of imagination. This deduction can go further to generalize every property of this solid: since surface and volume are mathematically linked in many cases, there is no doubt that one would think that the volume of the solid is infinite, but it is not always the case. The volume is finite!

This is what has been tormenting people since Torricelli discovered that solid. It is intuitively impossible to have such a solid, but mathematics has proved the contrary.

An Analysis of the Volume of the Solid

It has been shown how the surface area of this solid of curved surface can be infinite and the possibility of different measures of object based on the space considered. Now, it is shown how the volume is finite by two methods. When the function f(x) = 1/x from 1 to infinity is considered and is revolved around the x-axis, the volume formula of the Horn can be given by the Disk Method as

$$V = \pi \int_a^b [f(x)]^2 dx$$

Therefore, the volume of the Gabriel's Horn is

$$V = \pi \int_1^\infty \frac{1}{x^2} dx.$$

Although this integral is similar to a Riemann integral, it is not because both function and domain are unbounded. Some definitions about how to construct a Riemann integral are needed before going any further:

Definition. *If* $P = \{x_{0,} x_1, x_2, \dots, x_n\}$ *is a partition on the closed interval* $[a, b]$ *and f is a function defined on that interval, then the n-th Riemann Sum of f with respect to the partition P is*

$$R(f, P) = \sum_{j=1}^{n} f(t_i)(x_j - x_{j-1}),$$

where t_j is an arbitrary number in the interval $[x_{j-1}, x_j]$ (Wachsmuth, 2007, n.p.).

Definition. *The upper sum of f with respect to the partition P is*

$$U(f,P) = \sum_{j=1}^{n} c_j \left(x_j - x_{j-1}\right),$$

where c_j is the supremum of f(x) in the interval$[x_{j-1}, x_j]$, and the lower sum of f with respect to the partition P is

$$L(f,P) = \sum_{j=1}^{n} d_j \left(x_j - x_{j-1}\right),$$

where d_j is the infinum of f(x) in the interval $[x_{j-1}, x_j]$ (Wachsmuth, 2007, n.p.).

Definition. *Suppose f is a bounded function defined on a closed, bounded interval $[a, b]$. Define the upper and lower Riemann integrals, respectively, as*

$$I^*(f) = \sup\{U(f,P) \colon P\ a\ partition\ of\ [a,b]\}$$

$$I_*(f) = inf\{L(f,P) \colon P\ a\ partition\ of\ [a,b]\}.$$

If $I^*(f) = I_*(f)$, the function f is called Riemann integrable and the Riemann integral over the interval $[a, b]$ is denoted by $\int_a^b f(x)dx$ (Wachsmuth, 2007, n. p.).

Because the integral that defines the volume is not a Riemann integral, it will later be shown how this integral can be evaluated by the improper Riemann integral method.

For now, focus need not to be put on the notion of volume, but a separate study of the behavior of the integral in itself when it goes to infinity is to be taken into account. In a more technical language, it needs to be determined if the improper integral will converge or diverge; if it converges, to what number it converges. To do so, the Integral Test theorem for infinite series and the Cauchy Completeness Theorem are used, for which a rigorous demonstration will be given.

The Integral Test theorem states that "If *f* is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either converge or both diverge" (Larson, Hostetler, & Edwards, 2006, p. 617). Thus, if it is shown that the series converges, it is automatically shown that the improper integral converges; it will then be shown that the series $\sum_{n=1}^{\infty} \frac{1}{x^2}$ converges. In order to show the convergence of this series, the Cauchy theorem that applies to Cauchy sequences and its phrasing for series are necessary, and it will be shown how the series follows Cauchy convergence criterion. The theorem and proof is taken from Briger's (2007) book *Real Analysis: A Constructive Approach*:

Theorem: *Every Cauchy sequence converges to some real number. In other words, if* (a_k) *is a Cauchy sequence, then there exists an L such that* $(a_k) \rightarrow L$.

Proof. Using (a_k) we will define a consistent and fine family of real intervals and invoke the Completeness Theorem we've already proved to show the existence of L. For each integer $k > 0$, define the interval I_k as follows. Let N_k be an integer such that $|a_i - a_j| < 1/k$ whenever $i, j \geq N_k$. (Here we are using the assumption that (a_k) is a Cauchy sequence.) Let $I_k = \left[a_{N_k} - \frac{1}{k}, a_{N_k} + \frac{1}{k}\right]$ and $F = \{I_k | k = 1,2, ...\}$. We note that the length of I_k is $\frac{2}{k}$, so the family F is clearly fine. To prove consistency, suppose that we have the interval I_k as above and another such interval$I_q = \left[a_{N_q} - \frac{1}{q}, a_{N_q} + \frac{1}{k}\right]$. Either $N_k \leq N_q$ or $N_q \leq N_k$; suppose that $N_k \leq N_q$. Then $\left|a_{N_q} - a_{N_k}\right| < \frac{1}{k}$, or $a_{N_q} - 1 < a_{N_k} < a_{N_q} + \frac{1}{k}$. The left-hand inequality gives $a_{N_q} - 1/q < A_{N_q} < a_{N_k} + 1/q$, while the right-hand inequality gives $a_{N_k} - 1/k < a_{N_q} < a_{N_q} + 1/q$. Thus I_k and I_q intersect. The case $N_q \leq N_k$ is proved the same way. Since the family F is fine in consistent, there is a unique real number L common to all of its intervals.

This L should work because it is arbitrarily close to each of the a_{N_k}, and the a_{n_k} are arbitrarily close to all a_i for *i* and *k* big enough. All that's left is an application of the triangle inequality to formalize this. So, suppose we are given $\epsilon > 0$. Choose k so large that $1/k < \epsilon/2$, and suppose that $i \geq N_k$. Then $|a_i - L| \leq |a_i - a_{N_k}| + |a_{N_k} - L| \rightarrow |a_i - L| < 1/k + 1/k \leq \epsilon$ (p.104).

Now, we need to show that $\sum_{n=1}^{\infty} \frac{1}{x^2}$ is a Cauchy sequence in order to say whether or not it will converge. A formal definition is "The sequence (a_k) is called a Cauchy sequence if, for any $\epsilon > 0$ there is an integer $N_a(\epsilon) > 0$ such that $|a_i - a_j| \leq \epsilon$ whenever *i* and *j* $\geq N_a(\epsilon)$" (Bridger, 2007, p. 102). Therefore, if we have two series $U_z = \sum_{n=1}^{z} \frac{1}{x^2}$ and $U_w = \sum_{n=1}^{w} \frac{1}{x^2}$ with $z \geq w > 1$, we can say that

$$U_z - U_w = \sum_{n=w+1}^{z} \frac{1}{x^2} \leq \sum_{n=w+1}^{z} \frac{1}{x(x-1)}$$

$$= \frac{1}{w} - \frac{1}{z} \leq \frac{1}{w}.$$

If we pick an appropriate N such that $N \geq 1/\epsilon$ such that $|U_z - U_w| \leq \epsilon$, then $\sum_{n=1}^{\infty} \frac{1}{x^2}$ will be a Cauchy sequence (Bridger, 2007, p. 103). One needs to notice that the notions of sequence and series in the proof of the Cauchy theorem have been interchanged since the Cauchy series must follow the same convergence criterion.

Since $\sum_{n=1}^{\infty} \frac{1}{x^2}$ converges, $\int_1^{\infty} \frac{1}{x^2} dx$also has to converge because of the Integral Test theorem. Now that the integral converges, the integral is evaluated by the improper integral method:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{c\to\infty}\int_1^c \frac{1}{x^2}\,dx$$

$$= \lim_{c\to\infty}\left[\frac{-1}{x}\right]_1^c$$

$$= \lim_{c\to\infty}\left[\frac{-1}{c}+1\right] = 1.$$

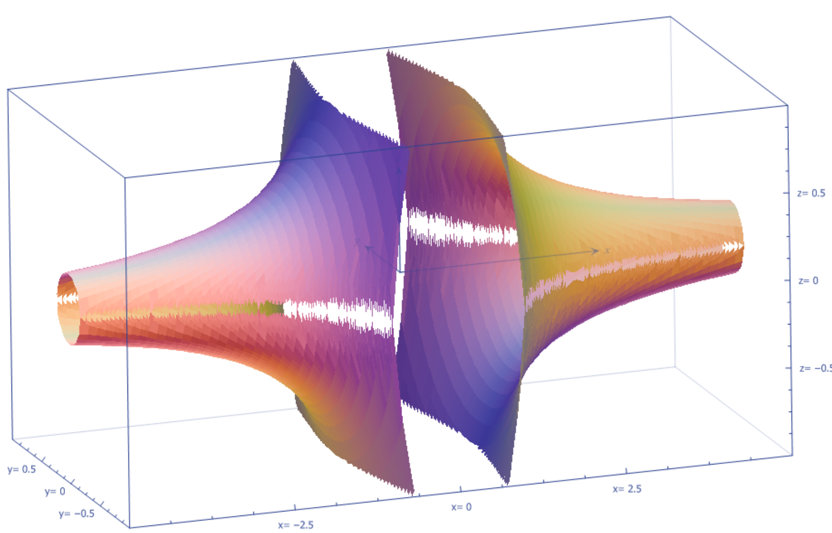
Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges to 1, the volume V of the Gabriel's Horn is thenπ.

For this method, instead of considering only the x and y-axes, the Horn is defined by adding the z-axis; thus, the function of the solid of revolution is now

$$z^2 + y^2 = \frac{1}{x^2}.$$

A graph to this function is a double-headed horn along the x-axis:

Figure 6



Since Gabriel's Horn is a solid of revolution given by $x^2 = \frac{1}{z^2+y^2}$, to determine its volume is more convenient with cylindrical coordinates. When $z = r\sin\theta$ and $y = r\cos\theta$ with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ on the *zy*-plane, the volume of one of the horns is

$$V = \frac{1}{2}\int_0^{2\pi}\int_0^{1}\int_0^{\frac{1}{r}} r\,dx\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{1} 1\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1\,d\theta$$

$$= \frac{1}{2}\,(2\pi) = \pi.$$

It can be seen that this result is exactly the same as that given by the Disk Method.

After we have presented the base of Euclidean geometry and its astounding importance in mathematics and many societies and after we have developed the non-Euclidean geometry which contradicts Euclid's fifth postulate, it is seen that a wealth of knowledge can be possible outside the realm of the highly venerated Euclid's system. The solid, while inconceivable with the idea that two parallel lines will always be equidistant, is well understood with the diametrically opposite idea. One then discovers a striking truth about mathematics, that it is not trapped in what it is called reality: a tangible, intuitive life. Based on logic, mathematics can go farther than our imagination can sometimes go and leave us in awe when it systematically oppugns the knowledge that we have so long considered as the truth. To the previous question about the earth and the pea, it is now left to the reader to figure out the answer.

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