Symmetry of Functions

About the y-axis: (The function is even)

\[ f(x) = f(-x) \]
Test:
Replace \( x \) by \(-x\), simplify and check the resulting equation against original to see if they are equivalent (the same).
If they are, \( f \) is symmetric about the \( y \)-axis.
Example: \( y = x^2 \); \( y = (-x)^2 \); \( y = x^2 \)

About the x-axis:
Note: For completeness only: A relation that is symmetric about the \( x \)-axis will not be a function since it will not pass the vertical line test.
Test:
Replace \( y \) by \(-y\), simplify and check the resulting equation against original to see if they are equivalent (the same).
If they are, \( x \)-relation is symmetric about the \( x \)-axis.
Example: \( x = y^2 \); \( x = (-y)^2 \); \( x = y^2 \)

About the Origin: (The function is odd)

\[ -f(x) = f(-x) \]
Test:
Replace \( x \) by \(-x\) and \( y \) by \(-y\), simplify and check the resulting equation against original to see if they are equivalent (the same).
If they are, \( f \) is symmetric about the \( origin \).
Example: \( y = x^2 \); \(-y = (-x)^2 \); \(-y = -x^2 \); \( y = x^3 \)

Parabolas (up/down opening functions):
All parabolas (quadratics, second degree polynomial relations) are symmetric about their own axis*.

Functions:
The axis is \( x = h \) where the \( h \) is the \( h \)-vertex \( (h, k) \).
If the parabola is in the form \( y = a(x - h)^2 + k \), use \( x = h \) as the line of symmetry.
If the parabola is in the form \( y = ax^2 + bx + c \), use \( x = -b/(2a) \) as the line of symmetry.

Relations that are not functions:
The axis is \( y = k \) where the \( k \) is the \( k \)-vertex \( (h, k) \).
If the parabola is in the form \( x = a(y - k)^2 + h \), use \( y = k \) as the line of symmetry.
If the parabola is in the form \( x = ay^2 + by + c \), use \( y = -b/(2a) \) as the line of symmetry.

Notes:
1) Functions (unlike relations in general) cannot be symmetric about the \( x \)-axis (they would fail the vertical line test).
2) Cubics, quartics, etc. may be symmetric, but not necessarily so…
   However, if they are one of the form, \( y = ax^n + k \), where \( n \) is even, they will be symmetric about the \( y \)-axis
   If they are one of the form, \( y = ax^n \), where \( n \) is odd, they will be symmetric about the \( origin \)
   If they are one of the form, \( x = ay^n + h \), where \( n \) is even, they will be symmetric about the \( x \)-axis (but not a function)
   If they are one of the form, \( x = ay^n \), where \( n \) is odd, they will be symmetric about the \( origin \) (and will be a function)
3) Terms:
   Zero degree polynomial functions are horizontal lines \( y = a \) \((a \text{ is a constant})\)
   First degree polynomial functions are lines \( y = mx + b \)…lines of the form \( y = mx \) are symmetric about the origin
   Second degree polynomial functions are also called quadratics or parabolas \( y = ax^2 + bx + c \)
   Third degree polynomial functions are also called cubics \( y = ax^3 + bx^2 + cx + d \)
   Fourth degree polynomial functions are also called quartics \( y = ax^4 + bx^3 + cx^2 + dx + e \)

*If a relation is of the more general quadratic form \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \), with the \( B \neq 0 \), then the axis(’) will be tilted. (It may or may not be a function…a tilted parabola, for example, may not pass the vertical line test). This more general quadratic form introduces many ramifications and is studied in a trigonometry course to some extent via a topic called "Rotation of Axis". A thorough study of this type could result in an entire textbook (or a big section) (I assume that it's already been done).